A PRE-HILBERT SPACE CONSISTING OF CLASSES OF CONVEX SETS

BY

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ABSTRACT

An equivalence relation is defined in the set of all bounded closed convex sets in Euclidean space E^n. The equivalence classes are shown to be elements of a pre-Hilbert space A^n, and geometrical relationships between A^n and E^n are investigated.

1. **Introduction.** With the operations of vector addition and scalar multiplication, the class $\mathscr{K}^n$ of all bounded convex sets in E^n forms a topological semigroup with scalar operators. If $K_1, K_2 \in \mathscr{K}$, and we write $K_1 \sim K_2$ when there exist centrally symmetric convex sets S_1, $S_2 \in \mathscr{K}^n$ such that

$$K_1 + S_1 = K_2 + S_2, \tag{1}$$

then $\sim$ is an equivalence relation in $\mathscr{K}^n$. The equivalence class containing a given set K is denoted by $[K]$ and is called an *asymmetry class.* In [3] it was shown that with the operations

$$\begin{aligned} \lambda[K] &= [\lambda K], \\ [K_1] + [K_2] &= [K_1 + K_2], \end{aligned} \tag{2}$$

the set A^n of asymmetry classes forms a normed vector space. The purpose of this note is to show that an inner product can be defined in A^n so that it becomes a pre-Hilbert space (i.e. an infinite-dimensional inner-product space over the real numbers which is not complete), and to investigate briefly some geometrical properties of K_1, $K_2 \in \mathscr{K}^n$ which imply that the corresponding classes $[K_1]$, $[K_2]$ are orthogonal in A^n.

The treatment follows similar lines to that of G. Ewald [2] except that he considers a different equivalence relation: his relation is defined by (1) with S_1, S_2 representing convex sets which are centrally symmetric in the origin. Consequently the corresponding equivalence classes are not invariant under translation.

2. **Asymmetry functions.** The *Steiner point* or *curvature centroid* $s(K)$ of a convex set $K \in \mathscr{K}^n$ may be defined (see [6]) by the relation

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$$s(K) = \int_{\Omega_n} uH(K,u)d\omega \bigg/ \int_{\Omega_n} (i.u)^2 d\omega, \tag{3}$$

where i is any fixed unit vector,

u is a typical unit vector,

Ω_n is the n-dimensional unit sphere in E^n, centred on the origin,

$d\omega$ is an element of surface area of Ω_n, and

$H(K,u)$ is the supporting function of the set K, that is,

$$H(K,u) = \sup_{x \in K} x \cdot y. \tag{4}$$

We shall require the following properties of $s(K)$:

$$\begin{aligned} s(\lambda K) &= \lambda s(K), \\ s(K_1 + K_2) &= s(K_1) + s(K_2) \end{aligned} \tag{5}$$

for all $\lambda \in R$ *and* K, K_1, $K_2 \in \mathscr{K}^n$.

(6) *If* $\mathscr{K}^n$ *is topologised by the Hausdorff metric,* [1, p. 34] *then* $s(K)$ *is a continuous function of* K.

Both (5) and (6) following immediately from the definition (3).

Now let $[K] \in A^n$ be any asymmetry class. Since $[K]$ contains all the translates of each of its members, we may choose a representative $K \in [K]$ with the property that $s(K) = o$ (the origin). For such a representative, write

$$a(K,u) = H(K,u) - H(K,-u). \tag{7}$$

Then $a(K,u)$, will be called an *asymmetry function.*

(8) $a(K_1,u) = a(K_2,u)$ *if and only if* $K_1 \sim K_2$.

For suppose (1) holds. Since $s(K_1) = s(K_2) = o$ by choice of K_1 and K_2, (5 implies $s(S_1) = s(S_2) = s$ (say), and then by (4),

$$H(S_i,u) = \bar{H}(S_i,u) + s \cdot u \qquad (i = 1,2)$$

where $\bar{H}(S_i,u)$ is the supporting function of S_i relative to its centre. Hence

$$\begin{aligned} H(K_1,u) + \bar{H}(S_1,u) + s \cdot u &= H(K_1,u) + H(S_1,u) \\ &= H(K_1 + S_1,u) \\ &= H(K_2 + S_2,u) \\ &= H(K_2,u) + H(S_2,u) \\ &= H(K_2,u) + \bar{H}(S_2,u) + s \cdot u, \end{aligned} \tag{9}$$

and similarly,

$$H(K_1, -u) + \bar{H}(S_1, -u) - s \cdot u = H(K_2, -u) + \bar{H}(S_2, -u) - s \cdot u .$$

Subtracting this latter relation from (9) and using the fact that, by central symmetry,

$$\bar{H}(S_i, u) = \bar{H}(S_i, -u), \qquad (i = 1,2)$$

we obtain

$$a(K_1, u) = a(K_2, u)$$

as was to be shown.

Conversely, if $a(K_1, u) = a(K_2, u)$ then it is simple to verify that (1) holds with S_1, S_2 defined by

$$H(S_1, u) = \tfrac{1}{2}(H(K_2, u) + H(K_2, -u)),$$

$$H(S_2, u) = \tfrac{1}{2}(H(K_1, u) + H(K_1, -u)),$$

and so $K_1 \sim K_2$.

The main consequence of (8) is that $a(K,u)$ is an *asymmetry class invariant* so that $a([K],u)$ may be properly defined by

$$a([K],u) = a(K,u) \text{ for any } K \in [K].$$

Thus we have established the existance of a 1-1 mapping between A^n and the class of all asymmetry functions $a([K],u)$. The next statement shows that this mapping is an isomorphism:

(10) *For all real λ and $[K]$, $[K_1]$, $[K_2] \in A^n$,*

$$a(\lambda[K],u) = \lambda a([K],u)$$

and

$$a([K_1] + [K_2], u) = a([K_1], u) + a([K_2], u).$$

The proof follows immediately from (2) and (5). It depends essentially upon the additivity property of $s(K)$, which explains why asymmetry functions must be defined relative to this point as origin.

The rest of this section is concerned with characterising asymmetry functions. This we can do completely in the case $n = 2$.

(11) *For each $[K] \in A^n$, the asymmetry function $a([K],u)$*

(*a*) *is homogeneous in u, that is,*

$$a([K], \lambda u) = \lambda a([K],u) \textit{ for all real } \lambda,$$

(b) *is a continuous function of u in E^n,*

(c) *satisfies*

$$\int_{\Omega_n} ua([K],u)d\omega = o.$$

(d) *(in the case $n = 2$) is a differentiable function of u on the circle $\Omega_2 : |u| = 1$, except possibly at an enumerable number of points, and the derivative is of bounded variation on Ω_2.*

Assertions (a) and (b) follow from the properties of support functions, (d) follows from the properties of convex functions given in [4], and (c) is a consequence of (3) and (7). We now come to the main theorem:

(12) THEOREM. *Let $f(u)$ be a function defined for $u \in E^2$. Then $f(u)$ is an asymmetry function (i.e. can be expressed in the form (7) for some $K \in \mathscr{K}^2$) if and only if it has the properties* (a), (b), (c) *and* (d) *of* (11).

A slight modification of the proof of Theorem 3 in [2] enables us to see that $f(u)$ is an asymmetry function if it has a continuous second derivative except possibly at the origin. In theorem (12), the conditions are necessary as well as being sufficient. The analogue of (12) for $n > 2$ dimensions is not known; it is difficult to see what would be the appropriate condition corresponding to (d). The necessity of the conditions in (12) is clear; to prove the sufficiency we need three lemmas, the first two of which establish conditions for a given function of a real variable to be expressible as the difference of two convex functions.

(13) LEMMA. *Let $\phi(x)$ be a continuous function of a real variable x in an interval $[a, b]$ with the property that $\phi'(x)$ exists at all points of $[a, b]$ with the possible exception of an enumerable set $G \subset [a,b]$. Suppose further, that $\phi'(x)$ is an increasing function of x where it is defined, i.e.*

$$\phi'(x_1) \geqq \phi'(x_2) \text{ for } x_1 \geqq x_2, \text{ and } x_1, x_2 \in [a,b] \setminus G.$$

Then $\phi(x)$ is a convex function of x in $[a,b]$.

Proof. Let x_0 be any point of $[a,b]$. If $x_0 \notin G$, let $m = \phi'(x_0)$, and if $x_0 \in G$, let m be chosen so that

$$\phi'(x) \leqq m \text{ for } x < x_0 \ (x \in [a,b] \setminus G)$$

$$\phi'(x) \geqq m \text{ for } x > x_0 \ (x \in [a,b] \setminus G).$$

In either case $\frac{d}{dx}(\phi(x) - mx) \geqq 0$ for all $x \in [a,b] \setminus G$ with $x > x_0$ and so, by lemma* of Hobson [4, p. 365],

$$f(x) \geqq f(x_0) + (x - x_0)m \quad \text{for} \quad x > x_0.$$

* I am indebted to Dr. B. Kuttner for drawing my attention to this lemma.

In a similar manner we can show that this same inequality holds also in the case $x < x_0$. We deduce that $y = f(x_0) + (x - x_0)m$ is a supporting line to the graph of $y = f(x)$ at x_0. Since x_0 is any point of $[a, b]$ this is sufficient to show that $f(x)$ is convex in $[a, b]$ and the lemma is proved.

(14) LEMMA. *Let $\phi(x)$ be a continuous function of a real variable x in an interval $[a, b]$ with $\phi(a) = 0$. Suppose that $\phi'(x)$ exists at all points of $[a, b]$ with the possible exception of an enumerable set G, and is bounded variation. Then $\phi(x)$ can be written as a difference*

$$\phi(x) = \psi_1(x) - \psi_2(x) \tag{15}$$

of two convex functions $\psi_1(x)$, $\psi_2(x)$ in $[a, b]$.

Proof. Since $\phi'(x)$ is bounded variation, it can be written in the form

$$\phi'(x) = \chi_1(x) - \chi_2(x)$$

where $\chi_1(x)$ and $\chi_2(x)$ are increasing functions of x defined in $[a, b] \setminus G$. Hence the integrals

$$\psi_i(x) = \int_a^x \chi_i(t)dt \qquad (i = 1, 2)$$

are defined and then (15) follows. Further $\chi_1(x)$, $\chi_2(x)$ are convex by lemma (13), since $\psi_i'(x) = \chi_i(x)$ is an increasing function of x in $[a, b] \setminus G$.

The remainder of the proof of the theorem is concerned with modifying the above procedure to apply to functions of a vector variable $u \in E^2$, instead of functions of a real variable x. We shall adopt the following notation. Let $f(u)$ be a function of $u \in E^2$, and let r, θ be the polar coordinates of u, i.e.

$$u = (r \cos \theta, r \sin \theta).$$

Then we shall write $f(u) = f(r, \theta)$ when we wish the display the coordinates of u explicitly. In this notation, the conditions of (11) become,

(a′) $f(r, \theta) = r f(1, \theta)$ for all r, θ,

(b′) $f(1, \theta)$ is a continuous function of θ,

(c′) $\displaystyle\int_0^{2\pi} f(1, \theta) \cos \theta \, d\theta = \int_0^{2\pi} f(1, \theta) \sin \theta \, d\theta = 0,$

(d′) $f(1, \theta)$ is a differential function of θ for $0 \leqq \theta \leqq 2\pi$, except possibly on an enumerable set $G \subset [0, 2\pi]$, and this derivative is of bounded variation.

(16) *Suppose $f(1, \theta) \geqq 0$ is a continuous convex function of θ for $\alpha \leqq \theta \leqq \beta$, where $\beta - \alpha \leqq \pi$. Then $f(u)$ is a homogeneous continuous convex function of u in the sector $S: \alpha \leqq \theta \leqq \beta$ of E^2.*

Proof. We must show that if $u_3 = \lambda u_1 + \mu u_2$, then

(17) $$\lambda f(u_1) + \mu f(u_2) \geqq f(u_3)$$

for all λ, $\mu \geqq 0$ and all $u_1, u_2 \in S$. Let r_i, θ_i be the polar coordinates of u_i $(i = 1,2,3)$. Since $f(1,\theta)$ (and therefore $f(u)$) is continuous in S, it will suffice to prove (17) in the special case $\theta_1 - \theta_3 = \theta_3 - \theta_2 = \phi \geqq 0$, say. A simple calculation shows that this implies that $\lambda/\mu = r_2/r_1$, so we may put $\lambda = r_2$, $\mu = r_1$ and then $r_3 = 2r_1r_2\cos\phi$. Hence

$$\begin{aligned}\lambda f(u_1) + \mu f(u_2) - f(u_3) &= r_1r_2 f(1,\theta_1) + r_1r_2 f(1,\theta_2) - 2r_1r_2\cos\phi\, f(1,\theta_3)\\ &\geqq r_1r_2(f(1,\theta_1) + f(1,\theta_2) - 2f(1,\theta_3))\end{aligned}$$

since $0 \leqq \phi \leqq \frac{1}{2}\pi$, so $0 \leqq \cos\phi \leqq 1$, and $f(1,\theta_3) \geqq 0$,

$$\geqq 0 \text{ since } f(1,\theta) \text{ is convex.}$$

Hence (17) is true in this special case and so (16) is proved.

The next statement is the "local form" of (16), and follows easily from it. Details of the proof are omitted.

(18) *If $f(1,\theta) \geqq 0$ is locally convex at $\theta = \theta_0$, then $f(u)$ is locally convex at each point of the line $\theta = \theta_0$, $r > 0$.*

We now proceed to the proof of the theorem. Since $f(1,\theta)$ is continuous and $f(1,\theta) = -f(1,\theta+\pi)$, we can choose the coordinate system in such a way that $f(1,0) = 0$, By (d'), $f'(1,\theta)$ (the derivative with respect to θ) exists in $[0,\pi]$ except possibly for an enumerable set G, and is of bounded variation. By lemma (14) we can write

$$f(1,\theta) = g_1(\theta) - g_2(\theta) \qquad 0 \leqq \theta \leqq \pi$$

where $g_i(\theta)$ are convex functions of θ. For any integer n, define

$$k(\theta + 2\pi n) = \begin{cases} g_1(\theta) - \theta g_1(\pi)/\pi + K & 0 \leqq \theta < \pi \\ g_2(\theta-\pi) - (\theta-\pi)\, g_2(\pi)/\pi + K & \pi \leqq \theta < 2\pi \end{cases}$$

where K is chosen sufficiently large that $k(\theta) \geqq 0$ for all θ. It is easily verified that $k(\theta)$ is continuous for all θ, is convex in each of the ranges

$$n\pi \leqq \theta \leqq (n+1)\pi,$$

and that

$$f(1,\theta) = k(\theta) - k(\theta+\pi).$$

Thus if we write $h(u) = rk(\theta)$ where $u = (r\cos\theta, r\sin\theta)$, we have

$$f(u) = h(u) - h(-u)$$

where $h(u)$ is continuous in E^2, and is convex in each of the half-planes $0 \leqq \theta \leqq \pi$, $\pi \leqq \theta \leqq 2\pi$ by (16).

To complete the proof we must express $f(u)$ as the difference of two functions which are convex in the *whole* plane. To do this, let $k_L^0, k_R^0, k_L^\pi, k_R^\pi$ be the left and right derivatives of $k(\theta)$ at $\theta = 0$, π as indicated, and put

$$R = \max(|k_L^0| + |k_R^0|, |k_L^\pi| + |k_R^\pi|).$$

Let T be the line segment joining the points $(R, \pi/2), (R, 3\pi/2)$ so that the supporting function of T is

$$H(T,u) = \begin{cases} rR \sin \theta, & 0 \leqq \theta \leqq \pi \\ -rR \sin \theta. & \pi \leqq \theta \leqq 2\pi \end{cases}$$

Since $H(T,u)$ is convex in E^2, the function

$$p(u) = h(u) + H(T,u)$$

is convex in each of the half-planes $0 \leqq \theta \leqq \pi$, $\pi \leqq \theta \leqq 2\pi$. Further, when $\theta = 0$, the left derivative of $p(1,\theta)$ is $k_L^0 - R$ and the right derivative is $k_R^0 + R$. Since, by the definition of R,

$$k_L^0 - R < k_R^0 + R,$$

we deduce that $p(1,\theta)$ is locally convex at $\theta = 0$, and so, by (18), $p(u)$ is locally convex along the line $\theta = 0$, $r > 0$. Similarly, considering $\theta = \pi$, we deduce that $p(u)$ is locally convex along the line $\theta = \pi$, $r > 0$, and we deduce that $p(u)$ is locally convex everywhere except possibly at the origin. But local convexity at the origin follows from the homogeneity of $p(u)$ and so $p(u)$ is convex in the whole plane. It may therefore be written as a supporting function $H(K,u)$ for a suitable convex set K. Further, from the definition of $p(u)$,

$$f(u) = H(K,u) - H(K, -u).$$

Finally,

$$o = \int_{\Omega_2} uf(u)d\omega = \int_{\Omega_2} uH(K,u)d\omega - \int_{\Omega_2} uH(K, -u)d\omega,$$

and so

$$\begin{aligned} o &= s(K) - s(-K), \\ &= 2s(K). \end{aligned}$$

Thus the Steiner point of K is at the origin, and Theorem (12) is proved.

3. **An Inner Product on A^n.** Let $[K_1], [K_2] \in A^n$, then define the inner product

$$(19) \qquad \langle [K_1] | [K_2] \rangle_n = \left(\frac{2}{\pi}\right)^n \int_{\Omega_n} a([K_1],u)\, a([K_2],u) d\omega.$$

This is, within a scalar factor, the usual inner product defined on the set C of all continuous functions on Ω_n. By the remark following theorem (12) it is clear that the set of all asymmetry functions is dense in the set

$$C^* = \{f \mid f \in C, \int_0^{2\pi} \sin\theta f(\theta) d\theta = \int_0^{2\pi} \cos\theta f(\theta) d\theta = 0\}$$

so that A^n is isomorphic to a dense subset of C^*. We can now show that

(20) *A^n is not complete.*

If $n = 2$, this result is an immediate consequence of theorem (12) for it is easy to construct a Cauchy sequence of asymmetry functions which converges to a function whose first derivative is not bounded variation. For general n the statement follows from:

(21) *Let $\mathscr{H}$ be a hyperplane ($(n-1)$-dimensional subspace of E^n) passing through the origin, and K_1, K_2 be convex sets in $\mathscr{H}$, then the inner products*

$$\langle [K_1] \mid [K_2] \rangle_{n-1}$$

of the corresponding asymmetry classes in $\mathscr{H}$, and

$$\langle [K_1] \mid [K_2] \rangle_n$$

of the corresponding asymmetry classes in E^n, are equal.

Hence A^{n-1} can be isometrically embedded in A^n, or alternatively, after identification, A^{n-1} may be regarded as a subspace of A^n. Thus A^2 is a subspace of A^n $(n > 2)$ and since A^2 is not complete, (20) follows immediately.

To prove (21), let $u \in E^n$ be any unit vector, and $u' \in \mathscr{H}$ be the unit vector whose direction is parallel to the perpendicular projection of u on $\mathscr{H}$. Then for any $K \subset \mathscr{H}$,

$$H(K, u) = |u \cdot u'| H(K, u').$$

This implies that the Steiner point of K, regarded as lying in $\mathscr{H}$ coincides with the Steiner point of K regarded as lying in E^n. Taking this point to be the origin,

$$a(K, u) = |u \cdot u'| a(K, u')$$

and so

$$\int_{\Omega_n} a([K_1], u) a([K_2], u) d\omega = \int_{-(1/2)\pi}^{(1/2)\pi} d\theta \int_{\Omega_{n-1}} a([K_1], u) a([K_2], u) \cos^2\theta \, d\omega$$
$$= \frac{\pi}{2} \int_{\Omega_{n-1}} a([K_1], u) a([K_2], u) d\omega$$

where $\Omega_{n-1} = \Omega_n \cap \mathscr{H}$. From this, and the definition (19), statement (21) follows.

It is of course, possible to define many norms in A^n other than that derived from the inner product (19). For example, if we put

$$\|[K]\| = \sup_{\Omega_n} |a([K],u)| \tag{22}$$

then C is complete, but theorem (12) enables us to construct a Cauchy sequence σ of asymmetry functions (in the norm (22)) which does not converge to an asymmetric function, and so A^n is still not complete. This statement answers a question raised in [3], for σ is a Cauchy sequence in the norm defined there, and so again A^n is not complete.

If $\mathscr{K}^n$ is topologised by the Hausdorff metric, and either of the norms (19) or (22) is defined on A^n, the natural mapping $\mathscr{K}^n \to A^n$ is continuous. This follows from the continuity of the supporting function and (6).

4. **Orthogonality.** We now consider some special geometric relationships between two convex sets $K_1, K_2 \in \mathscr{K}^n$ such that the classes $[K_1]$, $[K_2]$ are orthogonal in A^n with respect to the inner product (19).

(22) *Let T be any orthogonal transformation in E^n with the property that $T^2 = -I$. (Here $-I$ is the mapping which sends each $x \in E^n$ into $-x$.) Then, for any K, the classes $[K]$ and $[TK]$ are orthogonal in A^n.*

Proof.

$$\begin{aligned}\langle [K] \mid [TK] \rangle &= \int_{\Omega_n} a([K],u)\, a([TK],u) d\omega, \\ &= \frac{1}{2}\left(\int_{\Omega_n} a([K],u)a([TK],u)d\omega + \int_{T\Omega_n} a([K],u)a([TK],u)d\omega \right), \\ &= \frac{1}{2}\left(\int_{\Omega_n} a([K],u)a([TK],u)d\omega + \int_{\Omega_n} a([-TK],u)a([K],u)d\omega \right), \\ &= 0,\end{aligned}$$

since the sum of the integrands is zero. We deduce that the inner product zero, and so (22) is established.

In the case $n = 2$, the only transformation T satisfying (22) is rotation through a right angle, so that orthogonality in A^2 is closely related to the concept of orthogonality in E^2.

(24) *Let R, S be absolutely perpendicular subspaces in E^n (so that $\dim R + \dim S = n$). Let K_R be any convex set symmetric in R (i.e. unchanged by reflection in R), and K_S be any convex set symmetric in S. Then $[K_R]$ and $[K_S]$ are orthogonal in A .*

We omit the proof since it is very similar to that of (22).

If $A^n(R)$ represents the subset of A^n consisting of those classes $[K]$ possessing a representative symmetric in the subspace R, then by [3, p. 15], $A^n(R)$ and $A^n(S)$ are directly complementary subspaces in A^n. With the inner product (19), (24) shows that these are orthogonal complements in A^n.

Let K^* be the reflection of K in R, so that $-K^*$ is the reflection of K in S. Then the identity

$$K = \tfrac{1}{2}(K + K^*) + \tfrac{1}{2}(K - K^*)$$

enables us to express each $[K] \in A^n$ as the sum of the elements $[\tfrac{1}{2}(K + K^*)] \in A^n(R)$ and $[\tfrac{1}{2}(K - K^*)] \in A^n(S)$ in these subspaces. In fact the mapping

$$[K] \to [\tfrac{1}{2}(K + K^*)]$$

is the orthogonal projection of $[K]$ on to the subspace $A^n(R)$, and similarly for the subspace S.

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